COMBINATORICA 7 (1) (1987) 77—85

A LOCAL CHARACTERIZATION OF THE JOHNSON SCHEME

J. I. HALL

Received 5 April 1985

Within the Johnson scheme $\mathcal{J}(m, d)$ we find the graph $K(m, d)$ of d-subsets of an m-set, two such adjacent when disjoint. Among all connected graphs, $K(m, d)$ is characterized by the isomorphism type of its vertex neighborhoods provided m is sufficiently large compared to d.

1. Introduction

The Johnson Scheme $\mathcal{J}(m, d)$ is the association scheme [2] of all d-subsets of an m-set, the pair (x, y) of d-subsets in relation $\mathcal{J}_i$, $0 \leqq i \leqq d$, provided $|x \cap y| = d-i$. Each relation is symmetric; so we get, for each i, a graph $J_i(m, d)$ whose vertices are the elements of $\mathcal{J}(m, d)$ with x and y adjacent precisely when $(x, y) \in \mathcal{J}_i$. In this article we shall focus on two of these graphs:

$$J(m, d) = J_1(m, d)$$

$$K(m, d) = J_d(m, d).$$

(Here K stands for Kneser; see [9].) It should be noted that if $m < 2d$ then $K(m, d)$ (and possibly some of the other $J_i(m, d)$ as well) has no edges. Indeed if $m < d$, then $\mathcal{J}(m, d)$ and all its associated graphs have no vertices at all.

For Γ a graph with adjacency relation $\sim$, we define the *local subgraph* Γ_x to be that subgraph induced by Γ on the vertices of $\{y | y \sim x\}$. If $\Gamma = K(m, d)$ then, for all $x \in \Gamma$, Γ_x is isomorphic to $K(m-d, d)$ as is readily seen. We are concerned here with proving a partial converse to this statement.

Theorem 1. *Let n and d be positive integers, and let Γ be a connected graph such that, for each $x \in \Gamma$, Γ_x is isomorphic to $K(n, d)$. Then there is a constant k_d such that Γ is isomorphic to $K(n+d, d)$ provided $n \geqq k_d$.*

We say that a connected graph Γ as in the hypothesis of Theorem 1 is *locally* $K(n, d)$. If we define a function k on the positive integers by letting $k(d)$ be the smallest positive, integral k_d for which the conclusion of Theorem 1 is valid, then all that we know of k is contained in our second theorem.

Partial support provided by NSF (USA), SERC (UK), ZWO (NL).
AMS subject classification (1980): 05 B 30

Theorem 2. (1) $k(1)=1$ *and* $k(2)=7$;
(2) *for* $d \geqq 2$, $2d+1 \leqq k(d) \leqq 3d+1$.

$k(1)=1$ trivially. That $2d+1 \leqq k(d)$, for $d \geqq 2$, is nearly trivial as $K(2d, d)$ is the disjoint union of $\frac{1}{2}\binom{2d}{d}$ copies of the complete graph on two vertices. Up to isomorphism there are exactly three connected graphs which are locally $K(6, 2)$ (see [3]), so $k(2)=7$ is a consequence of Theorem 2. It is conceivable that $k(d)=2d+1$ for all $d \geqq 3$, however the nice geometric properties of $\mathcal{J}(7, 3)$ (e.g., see [4]) makes $k(3) \neq 7$ an interesting possibility.

We prove the two theorems simultaneously by showing that Theorem 1 holds with $k_d=3d+1$. There seems further room for improvement, so many of our intermediate results are proved in greater generality than that demanded by our theorems. The proof emphasizes the importance of $J(m, d)$ in any study of $\mathcal{J}(m, d)$. The main tool in the proof is a second graph structure defined on the same point set as Γ which ultimately emerges as $J(n+d, d)$. (Neumaier [8] has recently completed the characterization of $J(n, d)$ as a distance regular graph by its parameters (see also Moon [7]), and various people have given local characterizations of certain $J(n, d)$ (see [1, 7]).

We first proved Theorem 1 in late 1977. The original proof was more clumsy than that given here, and the best bound for $k(d)$ found was $k(d) \leqq d^2+2d-1$. This was still strong enough to prove $k(2)=7$, leaving $n=5$ as the only unresolved case in the classification of locally $K(n, 2)$ graphs. This led us to prove (see [5]) that up to isomorphism there are exactly three connected graphs which are locally the Petersen graph $K(5, 2)$. (The study of locally $K(n, d)$ graphs is the general situation mentioned in the first paragraph of [5] and of which the work on locally Petersen graphs is a special case.) This in turn motivated [6] which among other things contains a complete classification of locally $K(n, 2)$ graphs for $n \geqq 6$. (The graphs $K(n, 2)$ are the complements of the triangular graphs $J(n, 2)$. $K(6, 2)$ is the $(2, 2)$-generalized quadrangle.)

The theorems may find application in the characterization of the Johnson scheme among the primitive association schemes and distance regular graphs. It can also be used to characterize alternating and symmetric groups (of sufficiently large degree) by centralizers of various of their elements (the initial motivation for the theorem). L. Babai in 1981 and A. E. Brouwer in 1984 urged that the older results be written up for publication. The product of their encouragement is the present better proof of a sharper result.

Our graph theoretic notation and terminology is fairly standard. All subgraphs we consider are meant to be induced subgraphs; in particular, we identify each graph with its vertex set. Adjacency in the graph Γ is usually denoted by $\sim$. We have already introduced the notation $\Gamma_x=\{y\in\Gamma \mid y\sim x\}$. More generally, for $\Sigma \subseteq \Gamma$,

$$\Gamma_\Sigma = \bigcap_{x\in\Sigma} \Gamma_x;$$

and sometimes when $\Sigma=\{a, b, c, \ldots, z\}$ we write $\Gamma_{abc\ldots z}$ for Γ_Σ. The cardinality $|\Gamma_x|$, for $x\in\Gamma$, will be referred to as the *valency* of x in Γ (rather than the more standard term—degree).

2. Some properties of $\mathcal{J}$, J, and K

We first present without proof some well-known and elementary properties of $J(n, d)$ and $K(n, d)$.

Proposition 2.1. *$J(n, d)$ is a distance transitive graph of diameter* $\min(d, n-d)$ *and cardinality* $\binom{n}{d}$. *x and y are at distance c in $J(n, d)$ precisely when $(x, y) \in \mathcal{J}_c(n, d)$. In particular each vertex x of $J(n, d)$ has exactly $\binom{n-d}{d}$ vertices at distance d from it.* ▮

Notice that $\mathcal{J}(n, d)$ hence $K(n, d)$ can be recovered from $J(n, d)$. If n is sufficiently large $J(n, d)$ can be recovered from $K(n, d)$.

Proposition 2.2. *Let $\Gamma \simeq K(n, d)$ with $n \geqq 2d+1$, and define a new graph $J(\Gamma)$ whose points are those of Γ with x and y adjacent in $J(\Gamma)$ precisely when $\Gamma_{xy} \simeq K(n-d-1, d)$. Then $J(\Gamma) \simeq J(n, d)$, and indeed $J(K(n, d)) = J(n, d)$.* ▮

Therefore in the case $\Gamma \simeq K(n, d)$ with $n \geqq 2d+1$ we can speak of $J(\Gamma)$, $\mathcal{J}_i(\Gamma)$ (for $0 \leqq i \leqq d$), and $\mathcal{J}(\Gamma)$ without confusion.

In the graph Σ a *geodesic* from x to y is a Σ-path of minimal length connecting x to y. The subgraph Δ of Σ is *geodesically closed* provided it contains all geodesics of Σ between pairs of its vertices.

Of great importance in this article are certain subgraphs of $K(n, d)$ called subspaces. A *labelling Φ of $\Gamma \simeq K(n, d)$ by X* is a map Φ of the vertex set of Γ to the set of d-subsets of the n-set X such that $g \sim h$ precisely when $\Phi(g) \cap \Phi(h) = \emptyset$. A *$\Phi$-subspace Δ* of Γ of *degree m* is a subgraph $\Delta \simeq K(m, d)$ such that $\Phi(\Delta)$ is all d-subsets of some m-subset of X. One hopes that the collection of Φ-subspaces of Γ is actually independent of labelling, that Δ is a Φ-subspace precisely when it is a Φ'-subspace for all labellings Φ'. $K(n, d)$ for $n \leqq 2d$ has so little structure that this is not to be expected, but the next lemma shows this to be the case for $n \geqq 2d+1$. A *subspace* of Γ (a *Γ-subspace*) is a subgraph which is a Φ-subspace for all labellings Φ.

Lemma 2.3. *Let $\Gamma \simeq K(n, d)$ where $n \geqq 2d+1$. Then every Φ-subspace of Γ is a subspace of Γ. More precisely, the subgraph $\Delta \simeq K(m, d)$ is a subspace of Γ if and only if either*

(i) *$m \geqq 2d$ and Δ is geodesically closed within $J(\Gamma)$ or*

(ii) *$m < 2d$ and Δ is the intersection of subspaces Δ' of degree at least $2d$.*

Remark. All Φ-subspaces of $\Gamma \simeq K(n, d)$ are subspaces of Γ precisely when the automorphism group of Γ is S_n.

Proof. The only statement that needs checking is that a $\Delta \simeq K(m, d)$ with $m \geqq 2d$ is a subspace of Γ when it is geodesically closed in $J(\Gamma)$. Fix a labelling Φ of Γ by X, and set $\Phi(g) = g'$ for each $g \in \Gamma$. Let

$$\Delta' = \bigcup_{g \in \Delta} g'.$$

We claim that any d-subset of Δ' is h' for some $h \in \Delta$; that is, Δ is a subspace of Γ.

Let I be a subset of Δ' of size at most d, and suppose that $I\subset g'\cup h'$; for some pair $g, h\in\Delta$. As $m\geqq 2d$, there is an $f\in\Delta$ with $f\sim h$ and g on a geodesic of $J(\Gamma)$ from f to h, so that $I\subseteqq f'\cup h'$. But then the geodesic closure of Δ in $J(\Gamma)$ forces Δ to contain all of the Φ-subspace Σ of degree $2d$ labelled by $f'\cup h'$. In particular, for some $e\in\Sigma\subseteqq\Delta$, $e'\supseteqq I$.

Now let h', for some $h\in\Gamma$, be a d-subset of Δ'; and let Σ be a subset of Δ of minimal size subject to

$$h'\subseteqq\bigcup_{g\in\Sigma} g'.$$

The result of the previous paragraph forces $|\Sigma|=1$; so $\Sigma=\{h\}$, and $h\in\Delta$ as claimed. ∎

Lemma 2.4. *Let $\Gamma\simeq K(n, d)$, and let Δ be a subspace of Γ of degree at least $2d+1$. Then any subspace of Δ is a subspace of Γ.*

Proof. This is immediate from 2.3. ∎

Lemma 2.5. *Let $\Delta\subseteqq\Gamma\simeq K(n, d)$; and suppose that, for each $x\in\Delta$, Δ_x is a degree m subspace of Γ where $m\geqq 2d$. Then Δ is a subspace of degree $m+d$.*

Proof. Let Φ be a labelling of Γ by X, and set $\Phi(g)=g'$ for each $g\in\Gamma$. Here by assumption $n\geqq 3d\geqq 2d+1$, so that by 2.3 the Φ-subspaces of Γ are the subspaces of Γ.

Choose $p\in\Delta$, and let the Γ-subspace Δ_p of Γ be labelled by the subset Y of X, $|Y|=m$. Set $Z=p'\cup Y$, so that $|Z|=d+m$. Let h', for some $h\in\Gamma$, be a d-subset of Z. Then as $m\geqq 2d$ it is possible to choose $r\sim s$ within Δ_p in such a way that $h'\subseteqq p'\cup r'$. But then as Δ_s is geodesically closed within $J(\Gamma)$ (by 2.3), $h\in\Delta_s$. Therefore Δ contains all of the subspace Σ of Γ labelled by Z. Σ has valency $\binom{m}{d}=$ $=|K(m, d)|$ and so is a connected component of Δ. Because $m\geqq 2d-1$, every vertex of $\Gamma-\Sigma$ is adjacent to some vertex of Σ; so we have $\Delta=\Sigma$, a subspace of degree $m+d$. ∎

3. Graphs which are locally $K(n, d)$

In this section we prove the two theorems started in section 1. In view of the remarks of that section, these theorems are immediate consequences of the following two propositions.

Proposition 3.1. *Let Γ be a connected graph which is locally $K(n, d)$ for $n\geqq 3d+1$ and $d\geqq 2$. Then, for every $x\sim y\sim z\not\sim x$ with $\Gamma_{xyz}\simeq K(n-d-1, d)$, Γ_{xz} is a degree $n-1$ subspace of Γ_x and Γ_z.*

Proposition 3.2. *Let Γ be a connected graph which is locally $K(n, d)$ for $n\geqq 3d-1$ and $d\geqq 2$; and assume that, for every $x\sim y\sim z\not\sim x$ with $\Gamma_{xyz}\simeq K(n-d-1, d)$, Γ_{xz} is a degree $n-1$ subspace of Γ_x and Γ_z. Then $\Gamma\simeq K(n+d, d)$.*

As the propositions suggest, we give some of our results in greater generality than required for a proof of the theorems. It seems possible that additional arguments along these lines would lower the upper bound for $k(d)$ further.

We say that Γ is *locally* $K(*, d)$ if for each x of Γ there is an $n(x)$ with $\Gamma_x \simeq K(n(x), d)$.

Lemma 3.3. *Let n and d be positive integers with $n \geqq 2d$, and let Γ be a non-empty connected graph which is locally $K(*, d)$. Then Γ is locally $K(n, d)$ if and only if there is an $x \in \Gamma$ with $\Gamma_x \simeq K(n, d)$.*

Proof. One direction is clear. For the other direction it suffices (by connectivity) to prove $\Gamma_y \simeq K(n, d)$ for $y \sim x$. If $\Gamma_y \simeq \Gamma(m, d)$, then $(\Gamma_y)_x \simeq K(m-d, d)$ while $(\Gamma_x)_y \simeq K(n-d, d)$. Thus $\binom{m-d}{d} = |\Gamma_{xy}| = \binom{n-d}{d}$, with $m \geqq 2d$ as $\Gamma_{xy} \neq \emptyset$. The result follows. ∎

Lemma 3.4. *Let Γ be locally $K(*, d)$, and let $x \sim y \sim z \nsim x$. Then Γ_{xz} is locally $K(*, d)$; indeed if $\Gamma_y \simeq K(n, d)$, then $\Gamma_{xyz} \simeq K(m, d)$ for some $m \geqq n-2d+1$.*

Proof. This is immediate. ∎

Proof of Proposition 3.1. As $n-d-1 \geqq (3d+1)-d-1 = 2d$, the connected component Δ of Γ_{xz} containing y is locally $K(n-d-1, d)$ by 3.3 and 3.4. Furthermore $\Gamma_{xyz} \simeq K(n-d-1, d)$ is a Γ-subspace of the Γ_y-subspace $\Gamma_{xy} \simeq K(n-d, d)$. As $n-d \geqq 2d+1$, by 2.4 Γ_{xyz} is also a Γ_x-subspace of the Γ_x-subspace Γ_{xy}. Thus within Γ_x we may apply 2.5 to conclude that $\Delta \simeq K(n-1, d)$ is a subspace of Γ_x. Now $n-1 \geqq 2d-1$ implies each vertex of $\Gamma_x - \Delta$ is adjacent to some vertex of Δ, so $\Delta = \Gamma_{xz}$. ∎

For the rest of this section we are mainly pursuing Proposition 3.2., and so we adopt the following hypothesis throughout (although note that in 3.5 we relax the restriction on n to $n \geqq 2d+2$ or $(n, d) = (5, 2)$).

Hypothesis. Γ is a connected, locally $K(n, d)$ graph for integers n and d with $n \geqq 3d-1$ and $d \geqq 2$. For every $x \sim y \sim z \nsim x$ with $\Gamma_{xyz} \simeq K(n-d-1, d)$, Γ_{xz} is a degree $n-1$ subspace of Γ_x and Γ_z.

If x and z are as in the hypothesis, we write $x{:}z$, and we let $J(\Gamma)$ be the graph with vertex set Γ and x adjacent to z precisely when $x{:}z$. Notice that $J(\Gamma)$ induces $J(\Gamma_x) \simeq J(n, d)$ on each Γ_x. The import of the second sentence of the hypothesis is the equivalence for Γ of the following with $c=1$:

(i) Γ_{xz} is a degree $n-c$ subspace of Γ_x and Γ_z;
(ii) x and z have distance c in $J(\Gamma)$;
(iii) for some $y \in \Gamma_{xz}$, $(x, z) \in \mathcal{J}_c(\Gamma_y)$;
(iv) for all $y \in \Gamma_{xz}$, $(x, z) \in \mathcal{J}_c(\Gamma_y)$.

The main step in the proof of 3.2 is the observation (in 3.6) that for Γ (i)—(iv) are equivalent for any other constant c, $0 \leqq c \leqq d$. (This is trivially true for $c=0$.) Once this equivalence is proven, it can be checked that $J(\Gamma)$ has the same parameters as the distance regular graph $J(n+d, d)$, so the characterization by Neumaier [8] could be quoted to complete the proof of 3.2. We instead take the more elementary route of constructing a labelling which exhibits $\Gamma \simeq K(n+d, d)$. It is annoying that in order to verify the construction we must in addition to 3.6 use again the assumption $n \geqq 3d-1$. This is particularly unfortunate as it seems likely that 3.2 is valid with the inequality $n \geqq 3d-1$ replaced by $n \geqq 2d+1$.

Lemma 3.5. *Assume the hypothesis under the relaxed assumption* $n \geqq 2d+2$ *or* $(n, d)=(5, 2)$.

(1) *Let* $\pi=\{x_0:x_1 \dots :x_c\}$ *be a path of length* $c \leqq d+1$ *in* $J(\Gamma)$. *Then* Γ_π *contains* Δ, *a non-empty subspace of each* Γ_p, $p \in \pi$, *of degree at least* $n-c$.

(2) *The diameter of* Γ *is* 2, *and the diameter of* $J(\Gamma)$ *is* d. $x \sim z$ *in* Γ *if and only if* x *and* z *are at distance* d *in* $J(\Gamma)$.

Proof. (1) For $c=0, 1$ part (1) is true by assumption. We first consider the case $n \geqq 2d+2$ and later indicate those changes required for $(n, d)=(5, 2)$. Assume (1) is true for all $b \leqq c-1$, and let π be as given. Let $x=x_0$, $y=x_{c-1}$, $z=x_c$.

For $\Sigma=\{x: \dots :y\}$ we have a $\Delta_1 \subset \Gamma_\Sigma$ with Δ_1 a subspace of Γ_p, $p \in \Sigma$, of degree at least $n-c+1$. As $y:z$, $\Delta=\Delta_1 \cap \Gamma_z=\Delta_1 \cap \Gamma_{yz}$ is a Γ_y-subspace of degree at least $n-c$. Then as Δ is a subspace of the Γ_z-subspace $\Gamma_{yz} \simeq \simeq K(n-1, d)$ and $n-1 \geqq 2d+1$, Δ is a subspace of Γ_z by 2.4. Similarly, letting $w=x_{c-2}$, Δ is a subspace of the Γ_w-subspace $\Gamma_{wy} \simeq K(n-1, d)$ and so is a subspace of Γ_w. In this manner, Δ is a subspace of each Γ_p, $p \in \pi$. As $c \leqq d+1$ and $n \geqq 2d+1$, Δ is non-empty. This gives (1), provided $n \geqq 2d+2$.

In all the above argument, the only difficulty for the case $(n, d)=(5, 2)$ is the use when $c=2$ of 2.4 to prove that Δ is a subspace of Γ_x and Γ_z. (Remember that $|K(5-3, 2)|=1$.) But in this case one sees within Γ_r, for any $r \in \Delta$, that either $x \sim z$ or $x:z$; so Γ_{xz} is a subspace of Γ_x and Γ_z. We may then take $\Delta=\Gamma_{xy} \cap \Gamma_{xz}= =\Gamma_{xz} \cap \Gamma_{yz}$, an intersection of Γ_x-subspaces and of Γ_z-subspaces.

(2) $J(\Gamma)$ is connected because each Γ_r is connected within $J(\Gamma)$ and r is connected to $s \sim r$ within $J(\Gamma_t)$, for any $t \in \Gamma_{rs}$.

By (1) any path in $J(\Gamma)$ of length $d+1$ lies within some Γ_x. As $J(\Gamma_x)$ has diameter d, we see that $J(\Gamma)$ has diameter at most d. A second application of (1) then says that any two vertices x and z of Γ lie within some Γ_y, so Γ itself has diameter 2.

Any pair $x \sim z$ can not be at distance less than d within any $J(\Gamma_y)$, so the diameter of $J(\Gamma)$ equals d exactly and adjacent vertices of Γ have distance d in $J(\Gamma)$. Conversely if x and z are at distance d in $J(\Gamma)$, then there is some y such that x and z are at distance d within Γ_y. But then within Γ_y we see that $x \sim z$. ∎

Lemma 3.6. *Let* $x=x_0$ *and* $z=x_c$ *be at distance* c *in* $J(\Gamma)$, *and let* $\pi=\{x_0:x_1: \dots :x_c\}$ *be a length* c *path in* $J(\Gamma)$ *connecting* x *and* z. *Then* $\Gamma_{xz}=\Gamma_\pi$ *is a degree* $n-c$ *subspace of each* Γ_p, $p \in \pi$. *In particular each geodesic of* $J(\Gamma)$ *from* x *to* z *lies entirely within* Γ_y, *for all* $y \in \Gamma_{xz}$.

Proof. First suppose that $c=d$, so that $x \sim z$ by 3.5.2. Therefore $\Gamma_{xz} \simeq K(n-d, d)$, while 3.5.1 guarantees a subspace $\Delta \simeq K(n-d, d)$ within Γ_π. We conclude in this case that $\Gamma_{xz}=\Gamma_\pi$, as required. This allows us to assume for the balance of the proof that $c<d$ (again using 3.5.2).

Let $y \in \Gamma_{xz}$. Within $J(\Gamma_y)$ x and z have distance at least c, so Γ_{xyz} is at best of degree $n-d-c \geqq (3d-1)-d-(d-1)=d$. On the other hand 3.5.1 gives a subspace $\Delta \subseteqq \Gamma_\pi$ which already locally has this degree, and so must be a connected component of Γ_{xz}. As Δ has degree $n-c \geqq (3d-1)-(d-1)=2d$, any $w \in \Gamma_x-\Delta$ is adjacent to at least one vertex of Δ, so $\Delta=\Gamma_{xz}=\Gamma_\pi$, as required. ∎

Remark. In 3.6 we have proven the equivalence for Γ of the statements (i)—(iv) above for any constant c with $0 \leqq c \leqq d$. Notice that the case $c=d$ requires only the slightly weaker hypothesis of 3.5. The equivalence of (ii)—(iv) for all appropriate c is also a consequence of 3.5. (For each $y \in \Gamma_{xz}$ one can find within Γ_x a $w \sim z$ such that x lies on a $J(\Gamma_y)$ geodesic from w to z.)

Lemma 3.7. *Let $\Sigma \subseteqq \Gamma$. For $x \in \Gamma_\Sigma$, let Δ be the smallest subspace of Γ_x which contains Σ. Then Δ is also a subspace of Γ_z, for every $z \in \Gamma_\Sigma$, and of necessity is the smallest subspace of Γ_z containing Σ.*

Proof. We induct on the distance c from x to z in $J(\Gamma)$. The result is trivial for $c=0$. Next assume the result is true for all $y \in \Gamma_\Sigma$ at distance $b \leqq c-1$ in $J(\Gamma)$ from x. Let $x=x_0: \dots : x_{c-1}: x_c=z$ be a geodesic in $J(\Gamma)$ from x to z, and set $y=x_{c-1}$. By 3.6 $\Sigma \subset \Gamma_y$, so by induction Δ is a subspace of Γ_y. Indeed Δ_y is a subspace of the Γ_y-subspace $\Gamma_{yz} \simeq K(n-1, d)$. If $n-1 \geqq 2d+1$, then because Γ_{yz} is a subspace of Γ_z, 2.4 proves Δ a subspace of Γ_z. Otherwise $(n\ d)=(5, 2)$, and the result follows easily. ∎

As a consequence of 3.7, any Γ_x-subspace Δ must now be a Γ_y-subspace as well, for any $y \in \Gamma_\Delta$; so Δ can be called a *subspace* of Γ without ambiguity. The Δ of 3.7 is the subspace *spanned by* Σ.

We are now ready to describe a labelling of the vertices of Γ by $\{1, \dots, n+d\}$ which exhibits the isomorphism of Γ and $K(n, d)$. We prove this in a sequence of lemmas.

To each g of Γ we associate a subset g' of the set $\{1, \dots, n+d\}$. Furthermore, for any subset Σ of Γ we define

$$\Sigma' = \bigcup_{g \in \Sigma} g';$$

so for instance we quickly see $\Gamma' = \{1, \dots, n+d\}$.

Choose an edge $p \sim q$ of Γ and label $\Gamma_p \cup \Gamma_q$ according to:

(i) $p' = \{1, \dots, d\}, \quad q' = \{n+1, \dots, n+d\}$;

(ii) $\Gamma'_p = \{d+1, \dots, n+d\}$,

$\Gamma'_q = \{1, \dots, n\}$,

$\Gamma'_{pq} = \{d+1, \dots, n\}$;

(iii) the labellings of Γ_p, Γ_q, and Γ_{pq} exhibit the isomorphisms $\Gamma_p \simeq K(n, d) \simeq \Gamma_q$ and $\Gamma_{pq} \simeq K(n-d, d)$.

Notice that if Δ is a subspace of $\Gamma_p \cup \Gamma_q$, then $|\Delta'|$ is the degree of Δ.

This still leaves unlabelled all $g \notin \Gamma_p \cup \Gamma_q$. For such a g we set

$$g' = \Gamma' - (\Gamma'_{gp} \cup \Gamma'_{gq}).$$

Lemma 3.8. (1) *For all $g \in \Gamma$, $g' = \Gamma' - (\Gamma'_{gp} \cup \Gamma'_{gq})$.*
(2) *For all $g \in \Gamma$, $g \sim x \in \Gamma_p \cup \Gamma_q$ if and only if $g' \cap x' = \emptyset$.*

Proof. (1) This is true by definition if $g \notin \Gamma_p \cup \Gamma_q$ and is clear if $g \in \Gamma_p \cap \Gamma_q$. Assume $g \notin \Gamma_p \cup \Gamma_q$. Then Γ_{gpq} has degree at least $(3d-1)-d-(d-1)=d$, so that $\Gamma'_{gq} = p' \cup \Gamma'_{gpq}$. Thus

$$\begin{aligned} g' &= \Gamma' - (\Gamma'_{gp} \cup p') \\ &= \Gamma' - (\Gamma'_{gp} \cup \Gamma'_{gpq} \cup p') \\ &= \Gamma' - (\Gamma'_{gp} \cup \Gamma'_{gq}). \end{aligned}$$

(2) Without loss of generality $x \in \Gamma_p$. As (2) is clear if $g \in \Gamma_p \cap \Gamma_q$, we assume $g \notin \Gamma_p \cap \Gamma_q$; so again Γ_{gpq} has degree at least d. We first claim that $\Gamma'_{gp} \supset \Gamma'_{gq} \cap \Gamma'_p$. Suppose $i \in \Gamma'_{gq} \cap \Gamma'_p$. Then $\{i\} \cup \Gamma'_{gpq} = \Delta'$, for some Γ_q-subspace Δ inside Γ_{gpq}. That is, $i \in \Gamma'_{gp}$ as claimed. Now by (1)

$$\begin{aligned} g' \cap x' = \emptyset \quad &\text{iff} \quad x' \subset \Gamma'_{gp} \cup \Gamma'_{gq} \\ &\text{iff} \quad x' \subset (\Gamma'_{gp} \cup \Gamma'_{gq}) \cap \Gamma'_p \\ &\text{iff} \quad x' \subset \Gamma'_{gp} \\ &\text{iff} \quad x \sim g \quad \text{by 3.7.} \ \blacksquare \end{aligned}$$

Lemma 3.9. *Let Δ_1 be a subspace of Γ_{gp}, and let Δ_2 be a subspace of Γ_{gq} with the degree of $\Delta_1 \cap \Delta_2$ at least d. Then the subspace of Γ_g spanned by Δ_1 and Δ_2 has degree $|\Delta'_1 \cup \Delta'_2|$.*

Proof. Both numbers equal $|\Delta'_1| + |\Delta'_2| - |\Delta'_1 \cap \Delta'_2|$. ∎

Lemma 3.10. *For all $g \in \Gamma$, $|g'| = d$.*

Proof. This is clear if $g \in \Gamma_p \cup \Gamma_q$, so we assume that g has distance a from p in $J(\Gamma)$ and distance b from q in $J(\Gamma)$ where $0<a<d$ and $0<b<d$. Thus $|\Gamma'_{gp}| = n-a \geqq (3d-1)-(d-1)=2d$,

$$|\Gamma'_{gq}| = n-b \geqq 2d, \quad \text{and}$$

$$|\Gamma'_{gpq}| \geqq n-a-b \geqq (3d-1)-(d-1)-(d-1) = d+1.$$

By 3.9 $|\Gamma'_{gp} \cup \Gamma'_{gp}|$ is the degree of the subspace of Γ_g spanned by $\Gamma_{gp} \cup \Gamma_{gq}$, so it is enough to prove that this spans Γ_g.

Choose $r \in \Gamma_{gpq}$. Then Γ_{gp} is spanned by $r \cup \Gamma_{gpr}$ as Γ_{gp} has degree at least $2d$. Similarly Γ_{gq} is spanned by $r \cup \Gamma_{gqr}$. Because $p \sim q$ within Γ_r, $(\Gamma_r)_{gp}$ and $(\Gamma_r)_{gq}$ span the subspace $(\Gamma_r)_g$ of degree $n-d$ in Γ_r. Next within Γ_g, the subspace spanned by $\Gamma_{gp} \cup \Gamma_{gq}$ is also spanned by $r \cup \Gamma_{gpr} \cup \Gamma_{gqr}$ and so by $r \cup \Gamma_{gr}$. That is, $\Gamma_{gp} \cup \Gamma_{gq}$ spans Γ_g, as required. ∎

Lemma 3.11. *If $g, h \in \Gamma$ with $g' = h'$, then $g = h$.*

Proof. Let $g, h \in \Gamma$ have $g' = h'$. By 3.8 this is the same as saying $\Gamma'_{gp} \cup \Gamma'_{gq} = \Gamma'_{hp} \cup \Gamma'_{hq}$ and $\Gamma_{gp} \cup \Gamma_{gq} = \Gamma_{hp} \cup \Gamma_{hq}$. If g' is contained in Γ'_{pq}, then $g, h \in \Gamma_{pq}$ by 3.8.1, whence $g = h$. Otherwise by 3.9 $\Gamma_g = \Gamma_h$, both being the span of $\Gamma_{gp} \cup \Gamma_{gq}$. For any $x \in \Gamma_g$, we find $g = h$ within Γ_x. ∎

Lemma 3.12. *If $|g' \cap h'| = d-1$, then $g{:}h$.*

Proof. This is certainly true either if $g, h \in \Gamma_p$ or if $g, h \in \Gamma_q$; so we may assume this is not the case. In particular $g' \cup h'$ meets both p' and q' non-trivially, so that Γ_{ghpq} has degree at least

$$(d+3d-1)-2d-(d-1) = d.$$

By assumption and 3.8

$$(\Gamma'_{gp} \cup \Gamma'_{gq}) \cap (\Gamma'_{hp} \cup \Gamma'_{hq}) = \Gamma'_{ghp} \cup \Gamma'_{ghq}$$

has size $n-1$, so by 3.9 $\Gamma_{ghp} \cup \Gamma_{ghq}$ spans a subspace Δ of degree $n-1$ inside Γ_{gh}. As $g \neq h$, this and 3.6 give $g:h$. ∎

Proof of Proposition 3.2. We first claim that $|\Gamma| = \binom{n+d}{d}$. Fix some $z \in \Gamma$. We count the vertices g of Γ according to the degree of Γ_{gz} which by 3.6 is a subspace of degree $n-c$ where c is the distance from g to z in $J(\Gamma)$, $0 \leqq c \leqq d$. Γ_z has $\binom{n}{c}$ subspaces of degree $n-c$. For each such subspace Δ there are $\binom{d}{c} = \binom{d}{d-c}$ distinct z for which $\Gamma_{gz} = \Delta$, as can be calculated using 3.6 within Γ_x, for any $x \in \Delta$. Therefore

$$|\Gamma| = \sum_{c=0}^{d} \binom{d}{d-c}\binom{n}{c} = \binom{n+d}{d}.$$

By 2.1 and 3.10—3.12 we see that $J(\Gamma)$ is isomorphic to $J(n+d, d)$ possibly with some additional edges. By 2.1 and 3.5.2 both $J(n+d, d)$ and $J(\Gamma)$ have diameter d and $\binom{n}{d}$ vertices at distance d from any fixed vertex. Thus the "distance d" graphs of these two graphs are isomorphic. That is, $K(n+d, d) \simeq \Gamma$. ∎

References

[1] A. Blokhuis and A. E. Brouwer, Locally 4-by-4 grid graphs, *preprint.*

[2] E. Bannai and T. Ito, *Algebraic Combinatorics* I, *Association Schemes*, Benjamin/Cummings, 1984.

[3] F. Buekenhout and X. Hubaut, Locally polar spaces and related rank three groups, *J. Algebra*, **45** (1977), 391—434.

[4] J. I. Hall, On identifying PG (3, 2) and the complete 3-design on seven points, *Annals of Discrete Mathematics*, **7** (1980), 131—141.

[5] J. I. Hall, Locally Petersen graphs, *J. Graph Theory*, **4** (1980), 173—187.

[6] J. I. Hall and E. E. Shult, Locally cotriangular graphs, *Geometriae Dedicata*, **18** (1985), 113—159.

[7] A. Moon, The graphs $G(n, k)$ of the Johnson schemes are unique for $n \geqq 20$, *J. Combin. Th.*, B, **37** (1984), 173—188.

[8] A. Neumaier, Characterization of a class of distance regular graphs, *J. Reine Angew. Math.*, **357** (1985), 182—192.

[9] A. Schrijver, Vertex-critical subgraphs of Kneser graphs, *Nieuw Archief voor Wisk.*, **26** (1978), 454—461.

J. I. Hall

Department of Mathematics
Michigan State University
East Lansing, Michigan 48824, USA